Qualitative Randomness

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This Talk

This talk is about one main idea: how to define algorithmic randomness within a qualitative probability structure.

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In particular it's quite useful for studying problems that Bayesians care about. So it would be nice to have reasons to say that algorithmic randomness is a natural (or at least justified) part of the Bayesian toolkit, the way standard probability theory is.

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We can make an analogy with probability—are agents' beliefs really well-represented by probability measures? Perhaps real agents are only capable of rational-valued credences, rather than arbitrarily precise real numbers.

This Talk

This latter issue has been addressed in the literature on qualitative probability. Briefly, this program shows that one can start with a relation of qualitative probability judgment, "A is at least as likely as B", and under certain conditions there is a (sometimes unique) probability measure that represents those judgments.

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Then we argue that this sort of relation is simple, intuitive, empirically verifiable, etc., and so the use of real numbers to represent belief is equally unobjectionable.

This is my target: to show that qualitative probability is already sufficient information not just for probability, but for algorithmic randomness.

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1. $\preceq$ is a total preorder with bottom element $\emptyset$ and top element Ω; and
2. if $B_1 \cap B_2 = \emptyset$ and if $A_1 \preceq B_1$ and $A_2 \preceq B_2$ then $A_1 \cup A_2 \preceq B_1 \cup B_2$.

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Then we call $\preceq$ a *qualitative probability*. If both $A \preceq B$ and $B \preceq A$ then we write $A \sim B$.

Definition

Let $\preceq$ be a qualitative probability on a σ-algebra $\mathcal{A}$. We say that $\preceq$ is *monotonely continuous* if given a monotone increasing sequence $A_n \uparrow A$ and an event B such that for all n, $A_n \preceq B$, then $A \preceq B$.

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Definition

If $\mathcal{A}$ is a σ-algebra and $\preceq$ is a monotonely continuous qualitative probability, then we call $(\mathcal{A}, \preceq)$ a *qualitative probability σ-algebra*.

Qualitative Probability

Definition

Let $(\mathcal{A}, \preceq)$ be a qualitative probability σ-algebra. We say that $(\mathcal{A}, \preceq)$ is *atomless* if for every $A \in \mathcal{A}$ such that $A \succ \emptyset$ there is $B \subset A$ such that $B \succ \emptyset$.

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The goal of qualitative probability is to provide conditions for the existence of a compatible probability measure, i.e., a measure P such that

$$A \preceq B \iff P(A) \leq P(B).$$

Qualitative Probability

Theorem (Villegas (1964))

If a qualitative probability σ-algebra $(\mathcal{A}, \preceq)$ is atomless then there is a unique compatible probability measure, and it is countably additive.

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Algorithmic Randomness

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Let $(\Omega, \mathcal{F}, \mu)$ be a computable probability space. A Martin-Löf test is a sequence $\{U_n\}_{n\in\omega}$ of c.e. open sets such that for all $n \in \omega$, $\mu(U_n) \leq 2^{-n}$.

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Definition

If $\{U_n\}_{n\in\omega}$ is a Martin-Löf test then $\bigcap_{n\in\omega} U_n$ is a Martin-Löf null set. We say that a point $x \in \Omega$ is Martin-Löf random if it is not in any Martin-Löf null set.

Algorithmic Randomness

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Let $(\Omega, \mathcal{F}, \mu)$ be a computable probability space. A Schnorr test is a sequence $\{U_n\}_{n\in\omega}$ of c.e. open sets such that for all $n \in \omega$,

1. $\mu(U_n) \leq 2^{-n}$; and
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Algorithmic Randomness

This is one of the standard templates for defining randomness notions: define a kind of sequential test that defines a null set, and say that a point is random if it is not in any such null set.

Algorithmic Randomness

But, crucially, these sequential tests are defined in terms of the measure assigned to the sets in the test.

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So recovering qualitative versions of algorithmic randomness notions is not as easy as rewriting the standard definitions in terms of a qualitative probability structure.

Measure Algebras

As it turns out, there is a way to recover randomness notions from a qualitative probability. The right path takes us through measure algebras.

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As it turns out, there is a way to recover randomness notions from a qualitative probability. The right path takes us through measure algebras.

This is a bit ironic since measure algebras don't have points, so how could there be *random* points?

Measure Algebras

Definition

A measure algebra is a pair $(\mathcal{B}, \mu)$ such that $\mathcal{B}$ is a Boolean algebra and μ is a probability measure on $\mathcal{B}$.

Moreover if $\emptyset$ is the only element of $\mathcal{B}$ which is assigned measure 0, then $(\mathcal{B}, \mu)$ is a complete metric space when equipped with the metric $d(A, B) = \mu(A\Delta B)$.

Measure Algebras

The most common way to construct a measure algebra is by taking a probability space and quotienting out the ideal of μ-null sets.

In other words, any two sets that differ only on a set of measure 0 are identified.

Computable Measure Algebras

Definition

A computable measure algebra is a triple $(\mathcal{B}, \mathcal{R}, \mu)$ such that

1. $\mathcal{B}$ is a σ-algebra;
2. $\mathcal{R}$ is a countable, computable Boolean algebra generating $\mathcal{B}$;
3. μ is a computable probability measure on $\mathcal{B}$.

Recovering Randomness from Measure Algebras

Definition

Let $\mathcal{B}$ be a Boolean algebra. An ultrafilter on $\mathcal{B}$ is a set $\mathcal{F} \subseteq \mathcal{B}$ such that

1. $\emptyset \notin \mathcal{F}$ and $\Omega \in \mathcal{F}$;
2. if $A \in \mathcal{F}$ then for any $B \supseteq A$, $B \in \mathcal{F}$;
3. if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
4. for all $A \in \mathcal{B}$, either $A \in \mathcal{F}$ or $\Omega \setminus A \in \mathcal{F}$.

Recovering Randomness from Measure Algebras

Definition

Let $(\mathcal{B}, \mathcal{R}, \mu)$ be a computable measure algebra. Suppose $\mathcal{G}$ is an ultrafilter on $\mathcal{B}$. We say that $\mathcal{G}$ is effectively generic if for any uniformly computable sequence $(A_n)_{n\in\omega}$ of elements of $\mathcal{B}$, $\bigcap_n A_n \in \mathcal{G}$.

Recovering Randomness from Measure Algebras

Theorem (Rute)

Let $(\mathcal{B}, \mathcal{R}, \mu)$ be the computable measure algebra obtained as the quotient of some computable probability space (Ω, μ). Then the following are equivalent for any $\mathcal{G} \subseteq \mathcal{B}$:

1. *$\mathcal{G}$ is an effectively generic ultrafilter;*
2. *$\mathcal{G}$ is an ultrafilter and $\bigcap_{cl} \mathcal{G} = \bigcap_{cpt} \mathcal{G} = \{x\}$ for some Schnorr random $x \in \Omega$.*

Recovering Randomness from Measure Algebras

So, Rute's theorem shows us that we can recover Schnorr randomness from computable measure algebras using ultrafilters. And, crucially, the ultrafilter definition doesn't make explicit mention of the probability measure.

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So, Rute's theorem shows us that we can recover Schnorr randomness from computable measure algebras using ultrafilters. And, crucially, the ultrafilter definition doesn't make explicit mention of the probability measure.

Now we need to translate this definition back into a qualitative probability framework. To do this we'll use some results from a different project of mine on computable qualitative probability.

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Computable Qualitative Probability

Definition

A computable qualitative probability σ-algebra is a tuple $(\mathcal{A}, \mathcal{R}, \preceq)$ where

1. $(\mathcal{A}, \preceq)$ is an qualitative probability σ-algebra;
2. $\mathcal{R} = \{R_i\}_{i \in \omega}$ is a computable countable Boolean algebra that generates $\mathcal{A}$;
3. for all $i \in \omega$,

$$(\leftarrow, R_i) = \{R_j \in \mathcal{R} \mid R_j \prec R_i\}$$
$$(R_i, \rightarrow) = \{R_j \in \mathcal{R} \mid R_i \prec R_j\}$$

are c.e. sets uniformly in i.

Computable Qualitative Probability

Theorem (JLW)

If $(\mathcal{A}, \mathcal{R}, \preceq)$ is an atomless computable qualitative probability σ-algebra then one can compute a unique computable probability measure $P : \mathcal{A} \to [0, 1]$ that is compatible with $\preceq$.

Computable Qualitative Probability

Definition

Let

$$U := \bigcup_{i \in I} R_i$$

where $R_i \in \mathcal{R}$ for all i, and suppose its complement $V = \Omega \setminus U$ is of the form

$$V := \bigcup_{j \in J} R_j$$

where again $R_j \in \mathcal{R}$ for all j. If $I, J \subseteq \omega$ are c.e. sets then we say that U, V are almost-computable events.

Computable Qualitative Probability

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In other words, they're the computable elements of the measure algebra $(\mathcal{A}, \mathcal{R}, P)$, where P is the measure compatible with the qualitative probability.

Computable Qualitative Probability

Definition

Let $\{U_n\}_{n\in\omega} = \{\bigcup_{i\in I_n} R_i\}_{n\in\omega}$ be a sequence of almost-computable events with complements $\{V_n\}_{n\in\omega} = \{\bigcup_{j\in J_n} R_j\}_{n\in\omega}$. If for all $n \in \omega$, the sets I_n, J_n are uniformly c.e., then we say that the $\{U_n\}_{n\in\omega}$ are uniformly almost-computable.

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One can show that this is the qualitative version of a sequence of events with uniformly computable measure.

Qualitative Randomness

Definition

Let $(\mathcal{A}, \mathcal{R}, \preceq)$ be a computable qualitative probability σ-algebra. Let $\mathcal{S}$ be an ultrafilter on $\mathcal{A}$. We say that $\mathcal{S}$ is a qualitative Schnorr ultrafilter if for any sequence of uniformly almost-computable events $\{U_n\}_{n\in\omega}$ in $\mathcal{A}$, $\bigcap_n U_n \in \mathcal{S}$.

Qualitative Randomness

Theorem (JLW)

Let $(\Omega, \mathcal{A})$ be a computable Polish space, and let $\preceq$ be a qualitative probability on $\mathcal{A}$ such that $(\mathcal{B}, \mathcal{R}, \preceq)$ is a quotient computable qualitative probability σ-algebra. Let P be the computable probability measure compatible with $\preceq$. Then the following are equivalent for any $\mathcal{S} \subseteq \mathcal{B}$:

1. *$\mathcal{S}$ is a qualitative Schnorr ultrafilter;*
2. *$\mathcal{S}$ is an ultrafilter and $\bigcap_{cl} \mathcal{S} = \bigcap_{cpt} \mathcal{S} = \{x\}$ for some P-Schnorr random $x \in \Omega$.*

Conclusion

This theorem shows us that we can recover Schnorr randomness from a computable qualitative probability structure.

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This theorem shows us that we can recover Schnorr randomness from a computable qualitative probability structure.

So we get a nice argument for randomness: if you think qualitative probability is sufficient to motivate the use of real numbers to represent belief, then you should also think it's sufficient motivation to use algorithmic randomness notions in epistemology.

Thank you!